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A universal programmable fiber architecture for the representation of a general incompressible linearly elastic material as a fiber-reinforced fluid

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ABSTRACT

Biological materials typically consist of elastic fibers immersed in an incompressible aqueous milieu. We consider the generality of an elastic material expressed as a fiber-reinforced incompressible fluid. We show that, in the linear regime, any (possibly inhomogeneous and/or anisotropic) incompressible elastic material can be represented as a collection of fifteen families of straight, parallel elastic fibers embedded in an incompressible medium. We can choose these fiber directions to correspond to the fifteen diagonals of an icosahedron that connect the midpoints of its antipodal edges. This fiber architecture, together with the incompressible medium in which it is immersed, is universal and programmable in the sense that its elastic constants can be chosen to model any linear incompressible elastic material, without having to adapt the fiber architecture to the actual microstructure of the material. An explicit algorithm is given to compute the local elastic constants for each fiber direction in terms of the local components of the elasticity tensor. Optimality properties of the icosahedral fiber architecture are conjectured, and numerical evidence in support of these conjectures is presented.

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1. Introduction

Biological tissue is typically wet and soft [1]. As a consequence of these two properties, the bulk modulus of biological tissue is typically much larger than its other elastic moduli, and this leads to the idealization known as an *incompressible* elastic material. It is also the case that biological tissue typically has an exquisitely complicated microstructure, often strongly oriented, with the local orientation differing from place to place within a given tissue, with the result that the elasticity of the tissue is both anisotropic and inhomogeneous [4,13]. Inasmuch as the microstructures of biological tissues come in a nearly infinite variety of types, one might think that a similarly large variety of mathematical models would be needed to capture this diversity of form.

In the present paper, we consider the class of incompressible, anisotropic, inhomogeneous linearly elastic materials. We first show that such a material may be completely characterized by a totally symmetric elasticity tensor: a tensor that is symmetric with respect to arbitrary permutations of its indices. This is in contrast to a general elasticity tensor, which is symmetric with respect to a restricted set of index permutations. This result implies in particular that the number of elastic moduli needed for incompressible linear elasticity is reduced to 15 from the 21 needed for general linearly elastic materials.

In the history of elasticity theory, there is an old controversy, reviewed by Love [10], as to whether the theory could be formulated in terms of 15 elastic parameters, as proposed by Cauchy on the basis of a particular model that involved only pairwise intermolecular forces acting along the line joining the centers of the two molecules in question, or whether 21 elastic parameters would be needed (see also [7]). This was decided by experiment on the elasticity of crystals in favor of the 21-parameter hypothesis. What the above result essentially shows, is that although Cauchy was wrong for general materials, his theory was general enough to encompass all incompressible materials, the greater generality of the 21-constant theory being irrelevant when one restricts consideration to incompressible deformation.

A particularly simple totally symmetric elasticity tensor is that of a family of straight parallel fibers. One may thus wonder whether a collection of such fiber families may suffice to express arbitrary incompressible linearly elastic materials.

To answer this question, we show that the 15 parameters needed to characterize an incompressible linearly elastic material may be chosen as the stiffness constants of 15 families of elastic fibers. What is significant here is that a *fixed* arrangement of fibers will suffice for the representation of *any* incompressible linearly elastic material, merely by adjusting the elastic constants of the fibers. (This adjustment may be made separately at each spatial location for the representation of an inhomogeneous material.) It is in this sense that the 15-family fiber architectures we describe are universal and programmable.

We first identify a class of fiber architectures that satisfy this property: the 15 directions in space obtained by taking the intersection of 6 distinct planes. Then, as a particular instance of such a choice of directions, we focus on the 15 space directions determined by the 30 midpoints of the *edges* of the regular icosahedron, which come in 15 antipodal pairs. Each such pair of antipodal midpoints defines a straight line and hence a direction for one of the fiber families. We argue that this configuration is the most symmetric possible, in a sense to be made precise, of all 15-direction fiber architectures that can model an arbitrary incompressible linearly elastic material.

Moreover, we derive explicit formulae, based in part on the discrete Fourier transform of order 5, that relate the traditional elastic moduli to the stiffnesses of the 15 families of fibers defined above. These formulae may be applied locally at each point of the material in the inhomogeneous case to determine the 15 stiffnesses of the 15 fibers (one member of each family) that intersect at the point in question.

As a particularly simple, but also a particularly interesting instance of the above theory, we note that a representation of an *isotropic* incompressible elastic material may be obtained by choosing the stiffnesses of the 15 families of fibers all equal to each other. One might think that the 15 fiber directions within such a material would be mechanically distinguishable from the other, non-fiber directions, but this is not the case.

The icosahedral configuration turns out not only to be the most symmetric possible, but also to possess optimality properties within the class of fiber families generated by 6 planes. We present computational evidence for this observation.

Let us make note of an alternative approach to express a linearly elastic material in terms of fiber families. Instead of fixing the fiber directions and varying only the strengths of the fiber families, we may vary both the directions and the strengths of the fiber families to suit the elastic material in question. A famous result in polynomial algebra implies that 6 fiber families are sufficient in this case. The proof of this striking result is unfortunately of a non-constructive nature, and an explicit computational determination of a suitable set of local fiber directions is difficult especially for inhomogeneous materials.

The notion of a “fiber-reinforced fluid” has been used before in modeling incompressible elastic materials in the context of the immersed boundary method [14], but it was always regarded there as a special case. (Although the immersed boundary method can handle general strain-energy functions, including those of nonlinear elasticity, the representation of immersed materials as collections of fibers is particularly simple and convenient.) What we show here is that this class of models (the fiber-reinforced fluids) is much more general than it seems. Indeed, it is completely general, i.e., universal and programmable, at the level of *linear* incompressible elasticity.

Extensive work has been done on the realizability of general elastic materials using specific microstructures [12,11,5]. In this context, our problem corresponds approximately to the problem of realizing an arbitrary incompressible linearly elastic materials by placing fibers in an incompressible medium. There is, however, a gap between such a physical picture of a “fiber-reinforced fluid” and the mathematical or computational notion set forth in this paper. Addressing this gap would be of great importance in understanding the mechanical properties of biological media, and is a direction for future research.

2. The number of elastic constants needed to characterize an incompressible linearly elastic material

We first review some classical material from linear elasticity [10,9]. A linearly elastic material is characterized by a strain-energy function W of the form

$$W = \frac{1}{2}\int C_{ijkl}(\mathbf{x})e_{ij}(\mathbf{x})e_{kl}(\mathbf{x})\,d\mathbf{x} \tag{1}$$

where $\mathbf{x} = (x_1, x_2, x_3)$ denotes position in Cartesian coordinates, $e(\mathbf{x})$ is the 3×3 strain tensor at position $\mathbf{x}$, and $C(\mathbf{x})$ is the $3 \times 3 \times 3 \times 3$ elasticity tensor at position $\mathbf{x}$. Here and throughout the paper (except where noted), the summation convention will be in effect for indices i, j, k and l. The dependence of C on $\mathbf{x}$ allows for the possibility that the material may be inhomogeneous. The strain $e(\mathbf{x})$ associated with a deformation that maps a point $\mathbf{x}$ to $\mathbf{x}'(\mathbf{x})$ is defined as follows:

$$e_{ij}(\mathbf{x}) = \frac{1}{2}\left(\frac{\partial x_i'}{\partial x_j}(\mathbf{x}) + \frac{\partial x_j'}{\partial x_i}(\mathbf{x})\right) - \delta_{ij} \tag{2}$$

where $\delta_{ij} = 1$ when $i = j$, and $\delta_{ij} = 0$ otherwise. The fundamental assumption of linear elasticity is that the matrix with elements $\partial x_i'/\partial x_j$ is close to the identity matrix. Thus, the e_{ij} are to be treated as small quantities.

Note that $e_{ij}(\mathbf{x}) = e_{ji}(\mathbf{x})$. It follows from this symmetry that there is no loss of generality if we impose the symmetry conditions

$$C_{ijkl}(\mathbf{x}) = C_{jikl}(\mathbf{x}) \tag{3}$$

and

$$C_{ijkl}(\mathbf{x}) = C_{ijlk}(\mathbf{x}). \tag{4}$$

Moreover, from the symmetry of the integrand in Eq. (1), it is clear that there is again no loss of generality if we impose the further symmetry

$$C_{ijkl}(\mathbf{x}) = C_{klij}(\mathbf{x}). \tag{5}$$

From now on, we assume that Eqs. (3)–(5) are indeed satisfied.

Even though the elasticity tensor $C(\mathbf{x})$ has 81 components, only 21 of these may be chosen independently, and then the rest are determined by the symmetry conditions stated above, Eqs. (3)–(5). One way to see that the correct count is 21 is to regard each of the pairs (i, j), (k, l) as a single index which, because of Eqs. (3)–(4), is allowed to take on one of only 6 distinct values, namely $(1, 1)$, $(2, 2)$, $(3, 3)$, $(1, 2) \equiv (2, 1)$, $(2, 3) \equiv (3, 2)$, $(3, 1) \equiv (1, 3)$. From this point of view the $C_{(i,j),(k,l)}$ are the elements of a 6×6 matrix, which is symmetric because of Eq. (5). A symmetric 6×6 matrix has $6 + (36 - 6)/2 = 21$ independent components.

It is important for the following considerations to note that $C(\mathbf{x})$ is not, in general, *totally symmetric*. In particular, we do *not*, in general, have the condition

$$C_{ijkl} = C_{ilkj} \tag{6}$$

which, together with Eqs. (3)–(5) would imply that one may interchange *any* two subscripts of $C(\mathbf{x})$ without affecting the value of the component in question. We shall, however, show that every elasticity tensor $C(\mathbf{x})$ is equivalent to another elasticity tensor $\tilde{C}(\mathbf{x})$ that *is* totally symmetric, where “equivalent” means that the two elasticity tensors give the same strain energy for all *incompressible* small deformations.

An incompressible deformation $\mathbf{x} \to \mathbf{x}'(\mathbf{x})$ is a deformation that obeys the constraint

$$\det\left(\frac{\partial \mathbf{x}'}{\partial \mathbf{x}}\right) = 1 \tag{7}$$

where $\det(A)$ denotes the determinant of A, and where $\partial \mathbf{x}'/\partial \mathbf{x}$ is the 3×3 matrix with ij element $\partial x'_i/\partial x_j$.

Within the framework of linear elasticity, the corresponding condition is that the strain matrix e has zero trace. Although this is well known, we give the proof for completeness. In components, Eq. (7) reads

$$\epsilon_{ijk} \frac{\partial x'_i}{\partial x_1} \frac{\partial x'_j}{\partial x_2} \frac{\partial x'_k}{\partial x_3} = 1 \tag{8}$$

where ϵ is the totally antisymmetric $3 \times 3 \times 3$ tensor with elements

$$\epsilon_{ijk} = \begin{cases} +1, & (i, j, k) = \text{even permutation of } (1, 2, 3), \\ -1, & (i, j, k) = \text{odd permutation of } (1, 2, 3), \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

Let ω be the antisymmetric 3×3 matrix defined by

$$\omega_{ij}(\mathbf{x}) = \frac{1}{2}\left(\frac{\partial x'_i}{\partial x_j}(\mathbf{x}) - \frac{\partial x'_j}{\partial x_i}(\mathbf{x})\right). \tag{10}$$

Combining this with Eq. (2), we see that

$$\frac{\partial x_i'}{\partial x_j}(\mathbf{x}) = \delta_{ij} + \omega_{ij}(\mathbf{x}) + e_{ij}(\mathbf{x}). \tag{11}$$

Thus, the constraint of incompressibility (Eq. (8)) becomes

$$\epsilon_{ijk}(\delta_{i1} + \omega_{i1} + e_{i1})(\delta_{j2} + \omega_{j2} + e_{j2})(\delta_{k3} + \omega_{k3} + e_{k3}) = 1. \tag{12}$$

In linear elasticity, both the strain matrix e and the rotation matrix ω are treated as small, first order quantities. Expanding Eq. (12) up to first order in ω and e, we get

$$\begin{aligned} 1 &= \epsilon_{ijk}\big[\delta_{i1}\delta_{j2}\delta_{k3} + (\omega_{i1} + e_{i1})\delta_{j2}\delta_{k3} + \delta_{i1}(\omega_{j2} + e_{j2})\delta_{k3} + \delta_{i1}\delta_{j2}(\omega_{k3} + e_{k3})\big] \\ &= 1 + \operatorname{trace}(\omega + e) \\ &= 1 + \operatorname{trace}(e) \end{aligned} \tag{13}$$

since ω is antisymmetric. It follows that the condition of incompressibility in linear elasticity, expressed in terms of the strain, is that

$$\operatorname{trace}(e) = 0. \tag{14}$$

We are now ready to prove

Theorem 1. *Given a* $3 \times 3 \times 3 \times 3$ *elasticity tensor C satisfying Eqs.* (3)–(5) (*but not necessarily Eq.* (6)), *there exists a unique tensor* $\tilde{C}$ *that satisfies not only Eqs.* (3)–(5) *but also Eq.* (6), *with the property that* $\tilde{C}$ *is* equivalent *to C in the sense that*

$$C_{ijkl}e_{ij}e_{kl} = \tilde{C}_{ijkl}e_{ij}e_{kl} \tag{15}$$

for all e such that $\operatorname{trace}(e) = 0$.

We note that equivalence in the sense above implies equality of the stress tensor in the following sense. Given an elasticity tensor C_{ijkl} and a traceless strain e_{kl}, one may compute the corresponding stress tensor as

$$\sigma_{ij} = C_{ijkl}e_{kl} - p\delta_{ij} \tag{16}$$

where δ_{ij} is the Kronecker delta and p is determined so that the incompressibility constraint is satisfied. From this, we see that two equivalent elasticity tensors C_{ijkl} and $\tilde{C}_{ijkl}$ will give rise to the same stress tensor if and only if

$$(C_{ijkl} - \tilde{C}_{ijkl})e_{kl} = q\delta_{ij} \tag{17}$$

is satisfied for any traceless strain e_{kl} and some scalar q. We will see that this is indeed the case in the course of proving the above result.

Before we present a proof of Theorem 1, we would like to do some dimension counting. The elasticity tensor has 21 independent components. What about a totally symmetric tensor? This question is equivalent to the combinatorial problem of choosing 4 objects of 3 types in which any type can be chosen as many times as one wishes. This can be computed as $\frac{(4+3-1)!}{4!(3-1)!} = 15$, see [19].

An elasticity tensor on the subspace of incompressible deformations defines a quadratic form on this subspace. This subspace has dimension $6-1=5$, and thus, the number of independent components of this quadratic form is equal to $(5\times 5-5)/2+5=15$ [2]. We thus have hope on dimensional grounds that the above theorem may be true.

We note that this result cannot be true for any spatial dimension other than 3. Suppose we consider the corresponding problem in dimension n. Let d_T be the number of degrees of freedom for a totally symmetric 4th order tensor, and d_S the number of degrees of freedom for a quadratic form on the space of trace free $n\times n$ matrices. We have [19]

$$d_T=\frac{(n+3)!}{4!(n-1)!}=\frac{n(n+1)(n+2)(n+3)}{24}, \tag{18}$$

$$d_S=\frac{n(n-1)(n+1)(n+2)}{8} \tag{19}$$

from which we see that $\frac{d_T}{d_S}=1-\frac{2(n-3)}{3(n-1)}$. For $n\geqslant 4$, $d_T<d_S$. In two dimensions, $d_T>d_S$, and thus, uniqueness fails. This also suggests that in two-dimensional incompressible elasticity a further reduction of the elasticity tensor beyond that to a totally symmetric tensor may be possible. We do not pursue the two-dimensional case here.

Proof of Theorem 1. Consider the vector space $\mathcal{U}$ of all tensors that satisfy the symmetry properties (3)–(5). This is a 21-dimensional vector space. We introduce two linear subspaces of $\mathcal{U}$. The subspace $\mathcal{V}$ is the linear space of totally symmetric tensors. This has dimension 15. We also introduce the subspace $\mathcal{W}$ of quadratic forms that evaluate to 0 for incompressible deformations.

We first show that an element of $\mathcal{W}$ can be written as

$$B_{ijkl}=b_{ij}\delta_{kl}+\delta_{ij}b_{kl} \tag{20}$$

for some 3×3 symmetric matrix b_{ij}. Denote by $\mathcal{B}$ the linear space of all tensors of the form of Eq. (20). We must show that $\mathcal{B}=\mathcal{W}$. It is easy to see that $\mathcal{B}\subset\mathcal{W}$. We just need to check that any tensor in $\mathcal{B}$ evaluates to 0 for incompressible deformations. We have

$$e_{ij}b_{ij}\delta_{kl}e_{kl}+e_{ij}\delta_{ij}b_{kl}e_{kl}=0 \tag{21}$$

since

$$\delta_{kl}e_{kl}=e_{ij}\delta_{ij}=\operatorname{trace}(e)=0. \tag{22}$$

To conclude that $\mathcal{B}=\mathcal{W}$, we show that the two spaces are of equal dimension. The matrix b_{kl} in Eq. (20) is a 3×3 symmetric matrix, from which we see that $\mathcal{B}$ is six-dimensional linear space. We thus show that $\mathcal{W}$ is also six-dimensional. The space of all symmetric 3×3 matrices is six-dimensional. Thus, a quadratic form on this space can be expressed as a symmetric 6×6 matrix $C_{\alpha\beta}$, where $1\leqslant\alpha,\beta\leqslant 6$. Express the quadratic form in a basis in which the last five of the six basis vectors span the traceless deformations. Any element in $\mathcal{W}$ evaluates to zero for any incompressible deformation, so $C_{\alpha\beta}=0$ for $2\leqslant\alpha,\beta\leqslant 6$. Since $C_{\alpha\beta}$ is a symmetric matrix, $C_{\alpha 1}=C_{1\alpha}$, and this means that $\mathcal{W}$ is a six-dimensional space.

We thus see that $\mathcal{W}=\mathcal{B}$ is a *six*-dimensional subspace of $\mathcal{U}$. The dimension of $\mathcal{V}$ and $\mathcal{W}$ sum to 21, the dimension of $\mathcal{U}$. If we can show that the intersection of the two linear spaces $\mathcal{V}$ and $\mathcal{W}$ is the zero tensor, the proof of the theorem is complete because this shows that all elements C_{ijkl} of $\mathcal{U}$ can be written uniquely as:

$$C_{ijkl}=\tilde{C}_{ijkl}+B_{ijkl},\quad \tilde{C}_{ijkl}\in\mathcal{V},\ \ B_{ijkl}\in\mathcal{W}. \tag{23}$$

Since B_{ijkl} vanishes on traceless deformations, Eq. (15) and the conclusion of the theorem follows. This shows incidentally that two equivalent elasticity tensors give rise to the same stress tensor in the sense of (17), since $B_{ijkl}e_{kl}$ is proportional to δ_{ij} for traceless e_{kl}.

We thus show that $\mathcal{V} \cap \mathcal{W} = \{\mathbf{0}\}$ where $\mathbf{0}$ is the zero tensor. Suppose $B_{ijkl} \in \mathcal{W}$ is totally symmetric, that is to say, $B_{ijkl} \in \mathcal{V}$. Then,

$$B_{1122} = b_{11}\delta_{22} + \delta_{11}b_{22} = b_{12}\delta_{12} + \delta_{12}b_{12} = B_{1212}. \tag{24}$$

We have $b_{11} + b_{22} = 0$. Likewise, we see that $b_{11} + b_{33} = b_{22} + b_{33} = 0$ and from this conclude that

$$b_{11} = b_{22} = b_{33} = 0. \tag{25}$$

Again, by total symmetry:

$$B_{1123} = b_{11}\delta_{23} + \delta_{11}b_{23} = b_{12}\delta_{13} + \delta_{12}b_{13} = B_{1213}. \tag{26}$$

We have $b_{23} = 0$. Likewise:

$$b_{23} = b_{13} = b_{12} = 0. \tag{27}$$

We thus see that $b_{ij} = 0$ for all ij and thus B_{ijkl} is the zero tensor.

We remark that the decomposition Eq. (23) can be computed explicitly. For any $C_{ijkl} \in \mathcal{U}$, determine b_{ij} (which in turn determines B_{ijkl}) as follows. For off diagonal elements of the symmetric matrix b_{ij}, let

$$b_{ij} = C_{ijkk} - C_{ikjk} \quad \text{(no summation)} \tag{28}$$

for i, j, k all different (note that this determines k as a function of the distinct numbers i and j). Determine the diagonal elements so that they satisfy

$$b_{ii} + b_{jj} = C_{iijj} - C_{ijij} \quad \text{(no summation)} \tag{29}$$

for $i \neq j$. □

Examination of the above proof reveals that there are two places where the fact that we are dealing with *three*-dimensional elasticity plays a role. The first is that the dimension of $\mathcal{V}$ and $\mathcal{W}$ add up to the dimension of $\mathcal{U}$. As remarked earlier, this can only happen for dimension 3. The second is in concluding that $\mathcal{V} \cap \mathcal{W} = \{\mathbf{0}\}$. A slight generalization of the above proof shows that this holds for any dimension greater than or equal to 3.

3. The elasticity tensor of a system of fibers

Consider a material in which is embedded a system of elastic fibers. In the reference configuration of the material, the fibers are relaxed and their geometry may be described by the direction field $\mathbf{u}(\mathbf{x})$, where $|\mathbf{u}(\mathbf{x})| = 1$, such that each of the fibers parametrized by arclength as $\mathbf{x} = \mathbf{X}(s)$ obeys a differential equation of the form

$$\frac{d\mathbf{X}}{ds} = \mathbf{u}\big(\mathbf{X}(s)\big). \tag{30}$$

Under the deformation $\mathbf{x} \to \mathbf{x}'(\mathbf{x})$ the fiber point $\mathbf{X}(s) \to \mathbf{X}'(s) = \mathbf{x}'(\mathbf{X}(s))$. The parameter s no longer measures arclength on the deformed fiber. Instead, we have

$$\begin{aligned}\left|\frac{d\mathbf{X}'}{ds}\right|^2 &= \frac{dX'_m}{ds}\frac{dX'_m}{ds}\\ &= \frac{\partial \mathbf{x}'_m}{\partial x_i}\frac{dX_i}{ds}\frac{\partial \mathbf{x}'_m}{\partial x_j}\frac{dX_j}{ds}\\ &= (\delta_{mi}+e_{mi}+\omega_{mi})(\delta_{mj}+e_{mj}+\omega_{mj})u_iu_j. \end{aligned}\tag{31}$$

To first order in e and ω, this becomes

$$\begin{aligned}\left|\frac{d\mathbf{X}'}{ds}\right|^2 &= \left[\delta_{ij}+2e_{ij}+(\omega_{ij}+\omega_{ji})\right]u_iu_j\\ &= 1+2e_{ij}u_iu_j\end{aligned}\tag{32}$$

where we have used the antisymmetry of ω and also that $|\mathbf{u}|=1$. Taking the square root of both sides (again, to first order in e), we get

$$\left|\frac{d\mathbf{X}'}{ds}\right| = 1+e_{ij}u_iu_j\tag{33}$$

and therefore

$$\left(\left|\frac{d\mathbf{X}'}{ds}\right|-1\right)^2 = e_{ij}e_{kl}u_iu_ju_ku_l.\tag{34}$$

If we assume that the fibers obey Hooke's law (for small strains), it follows that the elastic energy of the whole system of fibers may be written as

$$W=\frac{1}{2}\int S(\mathbf{x})u_i(\mathbf{x})u_j(\mathbf{x})u_k(\mathbf{x})u_l(\mathbf{x})e_{ij}(\mathbf{x})e_{kl}(\mathbf{x})\,d\mathbf{x}\tag{35}$$

where $S(\mathbf{x})$ is the fiber stiffness, which may be position dependent.

The above description pertains to what we will call a single family of fibers. Let us now generalize to the case of several such families of fibers, indexed by $n=1,2,\ldots,N$, each with its own direction field $\mathbf{u}^n(\mathbf{x})$. Then

$$W=\frac{1}{2}\int\sum_{n=1}^{N} S_n(\mathbf{x})u_i^n(\mathbf{x})u_j^n(\mathbf{x})u_k^n(\mathbf{x})u_l^n(\mathbf{x})e_{ij}(\mathbf{x})e_{kl}(\mathbf{x})\,d\mathbf{x}\tag{36}$$

and we see by comparison with Eq. (1) that the elasticity tensor of such a system comprised of several families of fibers is of the form

$$C_{ijkl}(\mathbf{x})=\sum_{n=1}^{N} S_n(\mathbf{x})u_i^n(\mathbf{x})u_j^n(\mathbf{x})u_k^n(\mathbf{x})u_l^n(\mathbf{x})\tag{37}$$

which is obviously totally symmetric, since any two of the indices i,j,k,l may be interchanged without affecting the value of C.

Before we proceed, we would like to clarify what we mean by a family of fibers. A system of fibers is defined by a continuous direction field and a strain energy related to that direction field given in (35). When considering several such fiber systems, all systems of fibers are assumed to move in the *same* continuous deformation field, which is constrained to be incompressible, by hypothesis. We shall make further comments on this notion at the end of the next section.

4. Representation of a totally symmetric elasticity tensor as the elasticity tensor of a system comprised of 15 families of fibers

Given a collection of fiber direction fields $\mathbf{u}^n(\mathbf{x})$, $|\mathbf{u}^n(x)| = 1$, $n = 1, \dots, N$, the relationship described by Eq. (37) between the totally symmetric elasticity tensor $C_{ijkl}(\mathbf{x})$ and the collection of fiber stiffnesses $S_n(\mathbf{x})$ is linear. Fixing the N fiber directions and varying $S_n(\mathbf{x})$, we see that the elasticity tensors of the N fiber families span a linear subspace of the space of totally symmetric tensors at each point $\mathbf{x}$. We would like to investigate when the elasticity tensors of these fiber families span the whole space $\mathcal{V}$ of all totally symmetric tensors. Since this question may be addressed for each $\mathbf{x}$ separately, we drop the argument $\mathbf{x}$ for now.

Before addressing this question, we make the following identification between symmetric tensors and *homogeneous polynomials*. This identification and the inner product we introduce later are standard material discussed in textbooks of multilinear algebra. We refer the reader especially to [16] where several issues pertinent to the present paper are discussed in detail. To any totally symmetric tensor C_{ijkl}, define the following homogeneous fourth degree polynomial in $\mathbf{t} = (t_1, t_2, t_3)$:

$$p_C(\mathbf{t}) = C_{ijkl} t_i t_j t_k t_l \tag{38}$$

which is a sum of $3^4 = 81$ terms. Consider the set of ordered triples of non-negative integers:

$$\mathcal{S} = \left\{ \mathbf{s} = (s_1, s_2, s_3) \in \mathbb{Z}^3 \;\middle|\; s_i \geqslant 0 \ (1 \leqslant i \leqslant 3),\ \sum_i^3 s_i = 4 \right\}. \tag{39}$$

The set $\mathcal{S}$ has 15 members. Given $\mathbf{s} \in \mathcal{S}$, let $\mathbf{i}(\mathbf{s}) = (i, j, k, l)$ be the combination of 4 integers between 1 and 3 such that 1 appears s_1 times, 2 appears s_2 times, and 3 appears s_3 times. For example:

$$\begin{aligned}
&\mathbf{i}(4,0,0) = (1,1,1,1),\\
&\mathbf{i}(2,1,1) = (1,1,2,3) = (1,1,3,2) = (1,2,1,3) = \cdots,\\
&\mathbf{i}(0,3,1) = (2,2,2,3) = (2,2,3,2) = (2,3,2,2) = (3,2,2,2).
\end{aligned} \tag{40}$$

We note that only the *combination* matters and not the order of the numbers (i, j, k, l). In fact, each $\mathbf{i}(\mathbf{s})$ can be expressed in

$$\gamma(\mathbf{s}) = \frac{4!}{s_1! s_2! s_3!} \tag{41}$$

different orderings of (i, j, k, l). With this notation in place, we may rewrite $p_C(\mathbf{t})$ as:

$$p_C(\mathbf{t}) = \sum_{\mathbf{s} \in \mathcal{S}} \gamma(\mathbf{s}) C_{\mathbf{i}(\mathbf{s})} \mathbf{t}^{\mathbf{s}}, \quad \mathbf{t}^{\mathbf{s}} = t_1^{s_1} t_2^{s_2} t_3^{s_3}. \tag{42}$$

This rewriting of $p_C(\mathbf{t})$ is possible because C_{ijkl} is totally symmetric and thus its value depends only on the combination (i, j, k, l) and not on its ordering. The above expression makes clear that there is a one-to-one correspondence between the space $\mathcal{V}$ of 4th order totally symmetric tensors in 3 dimensions and the space $\mathcal{P}_{34}$ of homogeneous 4th order polynomials in 3 variables. In particular, the dimension of both spaces is 15. As an example, let us examine $p_{\mathbf{u}}(\mathbf{t}) \in \mathcal{P}_{34}$ that corresponds to the fiber whose direction is given by $\mathbf{u}$.

$$p_{\mathbf{u}}(\mathbf{t}) = u_i u_j u_k u_l t_i t_j t_k t_l = (\mathbf{u} \cdot \mathbf{t})^4 \tag{43}$$

where $\mathbf{u}\cdot\mathbf{t}$ denotes the three-dimensional inner product. We shall call polynomials of the form $(\mathbf{u}\cdot\mathbf{t})^4$ *linear forms*.

We now rephrase the question posed at the beginning of this section in terms of homogeneous polynomials. Eq. (37) is equivalent to the following expression:

$$p_C(\mathbf{t}) = \sum_{n=1}^{N} S_n\big(\mathbf{u}^n\cdot\mathbf{t}\big)^4. \tag{44}$$

Thus, instead of asking whether the elasticity tensors of the fiber families span the space $\mathcal{V}$, we may ask the equivalent question of whether their linear forms span the space $\mathcal{P}_{34}$. We know that the space $\mathcal{V}$ (or equivalently $\mathcal{P}_{34}$) is 15-dimensional. Thus, N must be at least 15 in order for linear forms associated with the fiber directions to span $\mathcal{P}_{34}$.

It is well known that there are indeed unit vectors $\mathbf{u}^n$, $1\leqslant n\leqslant 15$, whose corresponding linear forms span $\mathcal{P}_{34}$ (see for example [16]). We shall say that such vectors form a $(3,4)$-*basis*. We start by giving an independent proof of the fact that $(3,4)$-bases exist.

In order to facilitate our study, we introduce an inner product on the space $\mathcal{P}_{34}$ [16]. Given two polynomials

$$p_C(\mathbf{t}) = \sum_{\mathbf{s}\in\mathcal{S}} \gamma(\mathbf{s}) C_{\mathbf{i}(\mathbf{s})}\mathbf{t}^{\mathbf{s}}, \qquad p_D(\mathbf{t}) = \sum_{\mathbf{s}\in\mathcal{S}} \gamma(\mathbf{s}) D_{\mathbf{i}(\mathbf{s})}\mathbf{t}^{\mathbf{s}}, \tag{45}$$

define the inner product $\langle\cdot,\cdot\rangle$ as:

$$\langle p_C, p_D\rangle = \sum_{\mathbf{s}\in\mathcal{S}} \gamma(\mathbf{s}) C_{\mathbf{i}(\mathbf{s})} D_{\mathbf{i}(\mathbf{s})} = C_{ijkl}D_{ijkl}. \tag{46}$$

This is just the natural component-wise inner product one would equip on the space of 4th order tensors, except that it is restricted to totally symmetric ones. What is of interest here is the following observation. Take the inner product of $p_C(\mathbf{t})$ with the linear form $(\mathbf{u}\cdot\mathbf{t})^4$:

$$\big\langle p_C, (\mathbf{u}\cdot\mathbf{t})^4\big\rangle = C_{ijkl}u_iu_ju_ku_l = p_C(\mathbf{u}). \tag{47}$$

Thus, the inner product of $p_C(\mathbf{t})$ with the linear form $(\mathbf{u}\cdot\mathbf{t})^4$ is nothing other than evaluation of $p_C(\mathbf{t})$ at $\mathbf{u}$.

Suppose we are given 15 vectors, $\mathbf{u}^1,\ldots,\mathbf{u}^{15}$. Denote by $\mathcal{Q}$ the linear space spanned by the corresponding linear forms $(\mathbf{u}^n\cdot\mathbf{t})^4$, $1\leqslant n\leqslant 15$. Showing that $\mathcal{Q}=\mathcal{P}_{34}$ is equivalent to saying that the orthogonal complement of $\mathcal{Q}$ must consist only of the zero polynomial which we denote by $\mathbf{0}$:

$$\mathcal{Q}^{\perp} = \big\{p\in\mathcal{P}_{34}\ \big|\ \big\langle p, \big(\mathbf{u}^n\cdot\mathbf{t}\big)^4\big\rangle = p\big(\mathbf{u}^n\big) = 0,\ 1\leqslant n\leqslant 15\big\} = \{\mathbf{0}\}. \tag{48}$$

We thus arrive at the following observation.

Lemma 1. *The vectors* $\mathbf{u}^1,\ldots,\mathbf{u}^{15}$ *form a* $(3,4)$*-basis if and only if*

$$p\in\mathcal{P}_{34}, \quad p\big(\mathbf{u}^n\big) = 0, \quad 1\leqslant n\leqslant 15, \tag{49}$$

implies that p *is the zero polynomial.*

We are now ready to state our first result. Take 6 planes $\pi_1, \dots, \pi_6$ in $\mathbb{R}^3$ that pass through the origin. Let $\mathbf{n}^1, \dots, \mathbf{n}^6$ be their corresponding unit normals. We say that the planes $\pi_1, \dots, \pi_6$, are in a *general position* if any vector triple $\mathbf{n}_p, \mathbf{n}_q, \mathbf{n}_r$, $1 \leqslant p < q < r \leqslant 6$, are linearly independent. Take any pair of planes π_p, π_q, $p \neq q$. Since the normals are linearly independent, the intersection is a line that passes through the origin. There are at most 15 such lines. These 15 lines are distinct if and only if the 6 planes are in a general position, as can be seen as follows.

Suppose the planes are in a general position. Take two pairs of planes, π_p, π_q and $\pi_{p'}, \pi_{q'}$. Let the corresponding intersection lines be ℓ_1 and ℓ_2, respectively. We want to show that $\ell_1 \cap \ell_2 = \mathbf{0}$. The set $\ell_1 \cap \ell_2 = \pi_p \cap \pi_q \cap \pi_{p'} \cap \pi_{q'}$ is the solution set of the following linear system in $\mathbf{z} \in \mathbb{R}^3$:

$$\mathbf{n}^\alpha \cdot \mathbf{z} = 0, \quad \alpha = p, q, p', q'. \tag{50}$$

Since three of the vectors $\mathbf{n}^\alpha$ are linearly independent, the only solution to the above linear system is $\mathbf{0}$. Conversely, if $\mathbf{n}_p, \mathbf{n}_q, \mathbf{n}_r$ is a linearly dependent triple, the three corresponding planes must meet at a line (or else, the three planes all coincide).

Proposition 1. *Take* 6 *planes* $\pi_1, \dots, \pi_6$ *in a general position. Then each pair of planes intersects in a line, and all* 15 *of these lines are distinct and go through the origin. We denote them* $\ell_1, \dots, \ell_{15}$. *Note that there are* 5 *such lines in each of the* 6 *planes. Let the unit vectors tangent to these lines* (*in either direction*) *be denoted by* $\mathbf{u}^1, \dots, \mathbf{u}^{15}$. *Then the linear forms corresponding to these unit vectors form a* $(3,4)$-*basis.*

Proof. Let $p \in \mathcal{P}_{34}$ be a polynomial such that $p(\mathbf{u}^n) = 0$ for all $1 \leqslant n \leqslant 15$. Take one of the planes π_1, and let $\ell_1, \dots, \ell_5$ be the five lines that lie on this plane. Take any point $\mathbf{z} \in \pi_1$ not at the origin. Since all the lines $\ell_1, \dots, \ell_5$ are distinct and go through the origin, one can find a line $\ell_{\mathbf{z}} \subset \pi_1$ that goes through $\mathbf{z}$ and intersects at each of the lines $\ell_1, \dots, \ell_5$ at distinct points, $\mathbf{z}_1, \dots, \mathbf{z}_5$. Since $p(\mathbf{u}^n) = 0$ and p is a homogeneous polynomial, it follows that $p(\mathbf{z}_n) = 0$, $1 \leqslant n \leqslant 5$. Let the unit direction vector corresponding to ℓ be $\mathbf{u}$. Consider the following polynomial in w:

$$p_{\mathbf{z}}(w) = p(\mathbf{z} + \mathbf{u}w). \tag{51}$$

This 4th degree polynomial evaluates to 0 at 5 points $w = w_i$ such that $\mathbf{z}_i = \mathbf{z} + \mathbf{u}w_i$. A 4th degree polynomial that has 5 distinct roots must be identically 0. Thus, $p_{\mathbf{z}}(w) \equiv 0$, and in particular, $p_{\mathbf{z}}(0) = p(\mathbf{z}) = 0$. Since $\mathbf{z}$ was an arbitrary point on π_1, we conclude that p is 0 at all points on π_1. Likewise, we conclude that p evaluates to 0 at all points on the planes $\pi_2, \dots, \pi_6$.

Now, consider any point $\mathbf{z}$ in $\mathbb{R}^3$ except the origin. One may again find a line ℓ that passes through $\mathbf{z}$ and intersects the planes $\pi_1, \dots, \pi_6$ at 6 distinct points. By considering the restriction $p_{\mathbf{z}}$ of p along ℓ as we did above, we conclude that the 4th degree polynomial $p_{\mathbf{z}}$ has 6 distinct roots since p evaluates to 0 at each of the 6 planes. Thus $p_{\mathbf{z}}$ is identically 0 and $p(\mathbf{z}) = 0$. Since $\mathbf{z}$ was an arbitrary point in $\mathbb{R}^3$ (except the origin), we conclude that p is 0 everywhere. By Lemma 1, the 15 vectors form a $(3,4)$-basis. $\square$

This simple construction of a set of $(3,4)$-basis vectors seems to be new. We generalize this construction to totally symmetric tensors or arbitrary order and dimension in Appendix A. We now have a large collection of sets of 15 fiber directions (each set being comprised of the pairwise intersections of 6 planes in general position) that we know form $(3,4)$-bases, and this, in particular, shows that there exists a large collection of sets of 15 fiber directions such that the elasticity tensors of the fibers span the space of totally symmetric elasticity tensors.

We next prove the following proposition, which states that "most" choices of the 15 fiber directions, in fact, form a $(3,4)$-basis.

Proposition 2. *Take* 15 *fibers of random orientation. That is to say, sample* 15 *times independently from the uniform probability measure on the unit sphere and take the unit vectors from the origin to these* 15 *points to be the fiber directions. With probability* 1, *these* 15 *directions form a set of* $(3,4)$-*basis vectors.*

Proof. For technical reasons, it is difficult to deal with sampling from a sphere. We shall instead sample the direction vectors from the punctured solid sphere:

$$D^3 = \left\{\mathbf{u} \in \mathbb{R}^3 \;\middle|\; 0 < |\mathbf{u}| < 1\right\}. \tag{52}$$

Proving the proposition with sampling from D^3 is equivalent to proving the above.

Let $\mathbf{u}^1, \ldots, \mathbf{u}^{15}$ be the 15 sampled directions. Linear independence of the 15 corresponding linear forms is equivalent to non-singularity of the following 15×15 Gram matrix U whose mn element is given by

$$U_{mn} = \left\langle \left(\mathbf{u}^m \cdot \mathbf{t}\right)^4, \left(\mathbf{u}^n \cdot \mathbf{t}\right)^4 \right\rangle = \left(\mathbf{u}^m \cdot \mathbf{u}^n\right)^4, \quad m, n = 1, \ldots, 15. \tag{53}$$

The determinant of U is a function that takes $(D^3)^{15}$ into $\mathbb{R}$. We would like to say that the set of points such that $\det(U) = 0$ (the zero set) has measure 0. Note that $\det(U)$ is a 120 degree homogeneous polynomial defined on an open set $(D^3)^{15}$ in $\mathbb{R}^{45}$. We know that $\det(U)$ is non-zero at points in $(D^3)^{15}$ that correspond to the 15 direction configurations of Proposition 1. Thus, $\det(U)$ is a non-zero polynomial. We shall now prove that the zero set of a non-zero polynomial has measure zero.

We prove this by contradiction. If the zero set has measure greater than zero, then there must be an open set $\mathcal{O}$ that is dense with points such that $\det(U) = 0$. Since $\det(U)$ is continuous, $\det(U)$ must be identically equal to 0 on $\mathcal{O}$. Take all 120 order partial derivatives of $\det(U)$ on $\mathcal{O}$. They all have to be 0 since $\det(U)$ is identically 0 on this set. Therefore, all coefficients of the monomials of $\det(U)$ are zero, and thus, $\det(U)$ must be identically 0 throughout $(D^3)^{15}$. This is a contradiction. $\square$

Proposition 1 gives us a concrete way to construct (3, 4)-bases, and Proposition 2 demonstrates the abundance of such configurations. We would now like to find an *ideal* configuration for purposes of representing a totally symmetric tensor in terms of linear forms.

Fix a certain 15 vector configuration $\mathbf{u}^1, \ldots, \mathbf{u}^{15}$ that forms a (3, 4)-basis. Given a totally symmetric tensor C_{ijkl}, we may see (37) or (44) (where $N = 15$) as linear equations in S_n:

$$C_{ijkl} = \sum_{n=1}^{15} S_n u_i^n u_j^n u_k^n u_l^n \quad \text{or equivalently} \quad p_C(\mathbf{t}) = \sum_{n=1}^{15} S_n \left(\mathbf{u}^n \cdot \mathbf{t}\right)^4. \tag{54}$$

The above constitute 15 linearly independent equations given that $\mathbf{u}^n$ form a (3, 4)-basis. Thus, for a given totally symmetric C_{ijkl}, the fiber strengths S_n are uniquely determined. We shall call (54) the *fiber representation* of the totally symmetric tensor C_{ijkl} with respect to $\mathbf{u}^1, \ldots, \mathbf{u}^{15}$. We may rewrite Eq. (54) by taking the inner product of $p_C(\mathbf{t})$ with respect to $(\mathbf{u}^m \cdot \mathbf{t})^4$:

$$p_C\left(\mathbf{u}^m\right) = \sum_{n=1}^{15} \left(\mathbf{u}^m \cdot \mathbf{u}^n\right)^4 S_n, \tag{55}$$

where we used (47). Note the appearance of the Gram matrix U, $U_{mn} = (\mathbf{u}^m \cdot \mathbf{u}^n)^4$ that we used in the proof of Proposition 2. We shall often find it convenient to use (55) instead of (54), especially because the Gram matrix U is invariant under three-dimensional rotations of the configuration $\mathbf{u}^1, \ldots, \mathbf{u}^{15}$.

Consider the problem of finding a fiber representation of a given incompressible linearly elastic material. It is desirable that our choice of the 15 vector configuration do not introduce an unwanted orientational bias in our fiber strengths S_n. One way to minimize such an orientational bias would be to have all fiber directions be "equivalent" to one another. We formulate this condition in the following way.

Let $\mathcal{A} = \{a^1, \ldots, a^{15}\}$ be a set of 15 different directions in $\mathbb{R}^3$. Any direction a^n corresponds to a pair of antipodal points on the unit sphere, $\mathbf{u}^n$ and $-\mathbf{u}^n$, or equivalently, a point on the real projective plane $\mathbb{P}^2(\mathbb{R})$.

We consider the subgroup G of three-dimensional rotations SO(3) that leaves the 15 directions invariant. That is to say:

$$G \equiv \{g \in \mathrm{SO}(3) \mid g\mathcal{A} = \mathcal{A}\}. \tag{56}$$

The expression ga^n denotes group action of g on the direction $a^n \in \mathcal{A}$. Then, $g\mathcal{A}$ denotes the set of directions that the elements of $\mathcal{A}$ are mapped to with the action of g.

The group G acts *transitively* on $\mathcal{A}$ if:

$$\forall a^n, a^m \in \mathcal{A}, \quad \exists g \in G \quad \text{s.t.} \quad ga^n = a^m. \tag{57}$$

This is a requirement that all vectors be equivalent geometrically with respect to rotations.

We can now state our result.

Theorem 2. *Let G act transitively on $\mathcal{A}$. Assume further that the* 15 *vectors corresponding to $\mathcal{A}$ form a* $(3,4)$-*basis. Then, $\mathcal{A}$ is a set of directions that corresponds to the* 15 *directions generated by straight lines that connect the antipodal midpoints of a regular icosahedron.*

Note that by duality of regular polyhedra we can substitute *dodecahedron* for *icosahedron* in the foregoing sentence, and the result will be exactly the same. We shall call this the *icosahedral configuration*.

Proof. We note that G is obviously a finite group. Since G acts transitively on $\mathcal{A}$, the order of G must be a multiple of the cardinality of $\mathcal{A}$. Thus, the order of G is divisible by 15.

It is well known that (see for example [3]) the finite subgroups of SO(3) are the cyclic groups of rotation around a certain axis, the dihedral groups, or the symmetry groups of regular polyhedra.

Suppose G is either cyclic or dihedral. Since the order of the group is divisible by 15, it must contain $2\pi/15$ rotations around a certain axis. Take any direction $a \in \mathcal{A}$ that does not coincide with the rotation axis. The rotations of a in multiples of $2\pi/15$ about the rotation axis generate the 15 distinct directions that comprise $\mathcal{A}$.

We now argue that these 15 directions do not form a $(3,4)$-basis. Let the 15 corresponding unit vectors be $\mathbf{u}^n$, $1 \leqslant n \leqslant 15$. These 15 directions are contained either in a double cone whose vertex is at the origin (in which case G is the cyclic group of order 15), or a plane passing through the origin (in which case G is the dihedral group of order 30). Suppose the 15 directions reside on a double cone, whose equation is given by $p(\mathbf{z}) = 0$, $\mathbf{z} \in \mathbb{R}^3$. Note that $p(\mathbf{z})$ is a homogeneous second degree polynomial such that $p(\mathbf{u}^n) = 0$ for all n. Now, consider the homogeneous 4th degree polynomial $p^2(\mathbf{z})$. Clearly, $p^2(\mathbf{u}^n) = 0$. Therefore $p^2 \in \mathcal{P}_{34}$ is a non-zero polynomial that evaluates to 0 at all points $\mathbf{u}^n$. By Lemma 1, the $\mathbf{u}^n$ do not form a set of $(3,4)$-basis vectors. Next, suppose the 15 directions lie on a plane, whose equation is given by $\pi(\mathbf{z}) = 0$, where $\pi(\mathbf{z})$ is a linear homogeneous polynomial. Similarly to the double cone case, $\pi(\mathbf{u}^n) = 0$ for all n, and $\pi^4 \in \mathcal{P}_{34}$ evaluates to 0 on all $\mathbf{u}^n$. Thus $\mathbf{u}^n$ do not form a $(3,4)$-basis.

Thus, G must be a symmetry group of one of the regular polyhedra. The icosahedral group is the only one whose order is divisible by 15, and thus G must be the icosahedral group.

Fix a regular icosahedral tessellation of the unit sphere (this is equivalent to specifying a unique inscribed regular icosahedron). Let G be the corresponding icosahedral group. Take any line that goes through the center of the unit sphere. By application of the 60 elements of G, we generate a family of line directions that is invariant under the action of G. The only way to obtain a 15 member family is to start with a line that goes through the midpoints of antipodal edges of the inscribed regular icosahedron.

We have now only to show that these 15 vectors form a set of $(3,4)$-basis vectors. Form 6 line segments by connecting the antipodal vertices of the regular icosahedron. Consider the 6 bisecting planes of these 6 line segments. The 15 line directions of the icosahedral configuration turn out to be the intersection lines of these 6 bisecting planes. By Proposition 1, these 15 directions form a set of $(3,4)$-basis vectors. □

We end this section by discussing the physical implications of the results we have obtained thus far. By the above results together with Theorem 1, we see that any incompressible elastic material has a representation as "fiber-reinforced fluid," i.e., as a system of elastic fibers embedded in an incompressible medium. Moreover, we see that a *fixed* arrangement of 15 families of fibers, each of which is comprised of straight parallel fibers running in one of the 15 directions defined above, will suffice for *all* incompressible elastic materials. There is no need to tailor the fiber geometry to the elasticity tensor: instead, one may simply adjust the fiber stiffnesses according to the prescription obtained by solving Eq. (58) at each point. We emphasize that this may be done separately at each point in the case of an inhomogeneous material. In the inhomogeneous case, it is particularly advantageous that the fiber directions are constant, since we therefore do not have to worry about the integrability of the fiber direction fields. In summary, then, we have defined a universal or "programmable" fiber architecture for the representation of an arbitrary incompressible linearly elastic material.

To avoid misunderstanding, we should make it clear that a fiber representation by itself does not enforce incompressibility. Instead, it is assumed that the fibers are embedded in an incompressible medium that only enforces the incompressibility constraint but does not otherwise contribute to the elasticity of the material.

Another related comment is that the above results on fiber representation can alternatively stand on their own without any reference to incompressibility. In this interpretation, these results say that any linearly elastic material that happens to be described by a totally symmetric elasticity tensor has a fiber representation of the type described above. Without the incompressibility constraint, the class of elastic materials that can be represented in this way, although quite large (15-dimensional) is not universal. The content of Theorem 1 is that this class becomes universal when we impose the incompressibility constraint.

Before we close this section, we briefly discuss the realizability of such a material. It is not difficult to implement such a fiber-reinforced incompressible medium *in silico*, a routine practice in the context of computational modeling in fluid-structure interaction. All fiber systems permeate three-dimensional space and interpenetrate one-another in an incompressible fluid. So long as the deformation field is a univalent function of space, all fiber systems will experience the same deformation. Thus, our mathematical notion of a fiber-reinforced fluid serves not only as a conceptual device for understanding incompressible elasticity but also as a useful modeling tool that is completely general in the linear regime.

In physical reality, different fiber systems cannot permeate all of space. Adjacent fibers may slide relative to one another thus violating the continuum hypothesis. Indeed, in such a case, non-local interactions are likely, as demonstrated in [5] in a different context, thus rendering invalid the description of the elastic material with an elasticity tensor. An approximate physical realization of our mathematical notion must thus be devised so as to prevent such relative motion.

5. Explicit computation of the fiber representation

In this section, we consider the problem of explicitly computing the fiber representation of a given totally symmetric tensor C_{ijkl} with respect to the icosahedral configuration. As noted in the previous section, this is equivalent to solving the linear system (55) which we reproduce here:

$$C^m \equiv p_C\left(\mathbf{u}^m\right) = \sum_{n=1}^{15}\left(\mathbf{u}^m \cdot \mathbf{u}^n\right)^4 S_n. \tag{58}$$

Below, we present a useful algorithm for solving this system. We first compute the matrix entries of the Gram matrix U, $U_{mn} = (\mathbf{u}^m \cdot \mathbf{u}^n)^4$.

The 15 icosahedral directions come in the form of 5 orthonormal triads, and it is therefore convenient to describe them by means of a double index, say (p,r), where $p = 1,\dots,5$ and $r = 1,2,3$. The specific formulae for the 15 directions that we consider are

$$\mathbf{u}^{(p,1)} = \left(\frac{A}{\sqrt{5}}\cos\frac{2\pi p}{5}, \frac{A}{\sqrt{5}}\sin\frac{2\pi p}{5}, \frac{1}{A}\right), \tag{59}$$

$$\mathbf{u}^{(p,2)} = \left(\frac{1}{A}\cos\frac{2\pi p}{5}, \frac{1}{A}\sin\frac{2\pi p}{5}, -\frac{A}{\sqrt{5}}\right), \tag{60}$$

$$\mathbf{u}^{(p,3)} = \left(-\sin\frac{2\pi p}{5}, \cos\frac{2\pi p}{5}, 0\right) \tag{61}$$

where

$$A = \frac{1}{\sqrt{\frac{1}{2}(1+\frac{1}{\sqrt{5}})}}. \tag{62}$$

Note that

$$\frac{A^2}{5} + \frac{1}{A^2} = 1. \tag{63}$$

With the help of this identity, it is easy to see that

$$\mathbf{u}^{(p,r)} \cdot \mathbf{u}^{(p,s)} = \delta_{rs}. \tag{64}$$

Thus, $\{\mathbf{u}^{(p,1)}, \mathbf{u}^{(p,2)}, \mathbf{u}^{(p,3)}\}$ is indeed an orthonormal triad. There is one such triad corresponding to each value of p.

It is important to note that $\mathbf{u}^{(p,r)}$ is periodic in p with period 5. It follows that we may choose any 5 successive integers for the domain of p. It will be convenient in the following to center these on zero. Thus we choose $p = -2, -1, 0, 1, 2$.

Our next task is to evaluate all of the matrix entries U_{mn}. We note that although the 15 icosahedral directions can only be determined up to three-dimensional rotations, the matrix entries only involve the inner product, and thus, do not depend on the particular set of icosahedral directions we chose in (59)–(62). In our present notation, these matrix entries are of the form $(\mathbf{u}^{(p,r)} \cdot \mathbf{u}^{(q,s)})^4$ and are readily evaluated as follows:

$$\left(\mathbf{u}^{(p,1)} \cdot \mathbf{u}^{(q,1)}\right)^4 = \left(\frac{A^2}{5}\cos\frac{2\pi}{5}(p-q) + \frac{1}{A^2}\right)^4, \tag{65}$$

$$\left(\mathbf{u}^{(p,2)} \cdot \mathbf{u}^{(q,2)}\right)^4 = \left(\frac{1}{A^2}\cos\frac{2\pi}{5}(p-q) + \frac{A^2}{5}\right)^4, \tag{66}$$

$$\left(\mathbf{u}^{(p,3)} \cdot \mathbf{u}^{(q,3)}\right)^4 = \left(\cos\frac{2\pi}{5}(p-q)\right)^4, \tag{67}$$

$$\left(\mathbf{u}^{(p,1)} \cdot \mathbf{u}^{(q,2)}\right)^4 = \left(\frac{1}{\sqrt{5}}\left(\cos\frac{2\pi}{5}(p-q) - 1\right)\right)^4, \tag{68}$$

$$\left(\mathbf{u}^{(p,1)} \cdot \mathbf{u}^{(q,3)}\right)^4 = \left(\frac{A}{\sqrt{5}}\sin\frac{2\pi}{5}(p-q)\right)^4, \tag{69}$$

$$\left(\mathbf{u}^{(p,2)} \cdot \mathbf{u}^{(q,3)}\right)^4 = \left(\frac{1}{A}\sin\frac{2\pi}{5}(p-q)\right)^4. \tag{70}$$

These formulae will be simplified later. What is important to note at this stage is that all of them are functions of $(p-q)$ and are periodic in $(p-q)$ with period 5.

In our present double-index notation, Eq. (58) reads as follows:

$$C^{(p,r)} = \sum_{q=-2}^{2}\sum_{s=1}^{3}\left(\mathbf{u}^{(p,r)}\cdot\mathbf{u}^{(q,s)}\right)^4 S_{qs}. \tag{71}$$

For each fixed value of p and q, let

$$\mathbf{C}(p) = \left(C^{(p,1)}, C^{(p,2)}, C^{(p,3)}\right)^T, \tag{72}$$

$$\mathbf{S}(q) = (S_{q1}, S_{q2}, S_{q3})^T \tag{73}$$

where ^{T} denotes the transpose operation, and let $M(p-q)$ be the 3×3 matrix with elements

$$\left(M(p-q)\right)_{rs} = \left(\mathbf{u}^{(p,r)}\cdot\mathbf{u}^{(q,s)}\right)^4 \tag{74}$$

where $r,s = 1,2,3$. It follows from this definition of M that

$$M(q-p) = \left(M(p-q)\right)^T \tag{75}$$

but it also follows from Eqs. (65)–(70) that

$$M(q-p) = M(p-q) \tag{76}$$

i.e., that $M(p-q)$ is an even function of its argument. Combining the above two results, we see that

$$M(p-q) = \left(M(p-q)\right)^T \tag{77}$$

i.e., that $M(p-q)$ is a symmetric matrix for each value of $(p-q)$.

Eq. (71) may now be rewritten as a discrete convolution equation:

$$\mathbf{C}(p) = \sum_{q=-2}^{2} M(p-q)\mathbf{S}(q) \tag{78}$$

where $p = -2,-1,0,1,2$, and where it is understood that $\mathbf{C}$ and $\mathbf{S}$, like M, are periodic with period 5. Such systems may be solved by the discrete Fourier transform, in this case of order 5, which together with its inverse, is defined as follows:

$$\hat{\mathbf{C}}(h) = \sum_{p=-2}^{2}\mathbf{C}(p)\exp\left(-i\frac{2\pi}{5}ph\right), \tag{79}$$

$$\mathbf{C}(p) = \frac{1}{5}\sum_{h=-2}^{2}\hat{\mathbf{C}}(h)\exp\left(+i\frac{2\pi}{5}ph\right) \tag{80}$$

with similar formulae for $\mathbf{S}$ and $\hat{\mathbf{S}}$, and also for M and $\hat{M}$. As is well known (and easily checked), the discrete Fourier transform reduces discrete convolution to multiplication. Thus, Eq. (78) becomes

$$\hat{\mathbf{C}}(h) = \hat{M}(h)\hat{\mathbf{S}}(h) \tag{81}$$

where $h = -2, -1, 0, 1, 2$. For each h, Eq. (81) is a 3×3 system of linear equations. The number of such systems that we actually need to consider is reduced by the following symmetry considerations.

Because $\mathbf{C}(p)$, $\mathbf{S}(q)$, and $M(p-q)$ are all real, it follows from Eq. (79) (and from its counterparts for $\mathbf{S}$ and M) that

$$\hat{\mathbf{C}}(-h) = \text{conjugate}\big(\hat{\mathbf{C}}(h)\big), \tag{82}$$

$$\hat{\mathbf{S}}(-h) = \text{conjugate}\big(\hat{\mathbf{S}}(h)\big), \tag{83}$$

$$\hat{M}(-h) = \text{conjugate}\big(\hat{M}(h)\big). \tag{84}$$

Therefore, the instances of Eq. (81) for $h = -2$ and $h = -1$ are merely the conjugates of the equations obtained by setting $h = +2$ and $h = +1$, respectively. We may therefore restrict consideration to $h = 0, 1, 2$. Moreover, in the evaluation of $\hat{M}(0)$, $\hat{M}(1)$, and $\hat{M}(2)$, we may take advantage of the even character of M to write

$$\hat{M}(h) = M(0) + 2\left(\cos\frac{2\pi}{5}h\right)M(1) + 2\left(\cos\frac{2\pi}{5}2h\right)M(2). \tag{85}$$

In particular,

$$\hat{M}(0) = M(0) + 2\big(M(1) + M(2)\big), \tag{86}$$

$$\hat{M}(1) = M(0) + 2\left(\cos\frac{2\pi}{5}\right)M(1) - 2\left(\cos\frac{2\pi}{10}\right)M(2), \tag{87}$$

$$\hat{M}(2) = M(0) - 2\left(\cos\frac{2\pi}{10}\right)M(1) + 2\left(\cos\frac{2\pi}{5}\right)M(2). \tag{88}$$

Now it follows from Eqs. (64) and (74) that

$$M(0) = I \tag{89}$$

where I is the 3×3 identity matrix. To proceed further in simplifying the above equations, it is helpful to know the following trigonometric formulae (which are related to the golden ratio and can be derived from elementary arguments concerning similar triangles, but which nevertheless seem to be not so well known as other formulae for sines or cosines of special angles):

$$\cos\frac{2\pi}{5} = \frac{-1+\sqrt{5}}{4}, \tag{90}$$

$$\cos\frac{2\pi}{10} = \frac{1+\sqrt{5}}{4}. \tag{91}$$

Making use of Eqs. (89)–(91), we may rewrite Eqs. (86)–(88) as follows:

$$\hat{M}(0) = I + 2\big(M(1) + M(2)\big), \tag{92}$$

$$\hat{M}(1) = I - \frac{1}{2}\big(M(1) + M(2)\big) + \frac{\sqrt{5}}{2}\big(M(1) - M(2)\big), \tag{93}$$

$$\hat{M}(2) = I - \frac{1}{2}\big(M(1) + M(2)\big) - \frac{\sqrt{5}}{2}\big(M(1) - M(2)\big). \tag{94}$$

To evaluate $M(1)$ and $M(2)$, we start from Eqs. (65)–(70) and set $p-q$ equal to 1 or 2, respectively. Again we make use of the trigonometric identities Eqs. (90)–(91) to express all results in terms of rational expressions involving only integers and $\sqrt{5}$. The results are

$$M(1)=\frac{1}{16}\begin{pmatrix} \frac{7+3\sqrt{5}}{2} & \frac{7-3\sqrt{5}}{2} & 1 \\ \frac{7-3\sqrt{5}}{2} & 1 & \frac{7+3\sqrt{5}}{2} \\ 1 & \frac{7+3\sqrt{5}}{2} & \frac{7-3\sqrt{5}}{2} \end{pmatrix}, \tag{95}$$

$$M(2)=\frac{1}{16}\begin{pmatrix} 1 & \frac{7+3\sqrt{5}}{2} & \frac{7-3\sqrt{5}}{2} \\ \frac{7+3\sqrt{5}}{2} & \frac{7-3\sqrt{5}}{2} & 1 \\ \frac{7-3\sqrt{5}}{2} & 1 & \frac{7+3\sqrt{5}}{2} \end{pmatrix}. \tag{96}$$

Substituting these results into Eqs. (92)–(94), we get

$$\hat{M}(0)=I+\frac{1}{16}\begin{pmatrix} 9+3\sqrt{5} & 14 & 9-3\sqrt{5} \\ 14 & 9-3\sqrt{5} & 9+3\sqrt{5} \\ 9-3\sqrt{5} & 9+3\sqrt{5} & 14 \end{pmatrix}, \tag{97}$$

$$\hat{M}(1)=I+\frac{1}{32}\begin{pmatrix} 3+\sqrt{5} & -22 & 3-\sqrt{5} \\ -22 & 3-\sqrt{5} & 3+\sqrt{5} \\ 3-\sqrt{5} & 3+\sqrt{5} & -22 \end{pmatrix}, \tag{98}$$

$$\hat{M}(2)=I-\frac{1}{8}\begin{pmatrix} 3+\sqrt{5} & -2 & 3-\sqrt{5} \\ -2 & 3-\sqrt{5} & 3+\sqrt{5} \\ 3-\sqrt{5} & 3+\sqrt{5} & -2 \end{pmatrix}. \tag{99}$$

It is remarkable that $\hat{M}(0)$, $\hat{M}(1)$, and $\hat{M}(2)$ are all of the form

$$\begin{pmatrix} 1+a & b & c \\ b & 1+c & a \\ c & a & 1+b \end{pmatrix} \tag{100}$$

of which the inverse, if it exists, is

$$\frac{1}{\Delta}\begin{pmatrix} (1+b)(1+c)-a^2 & ac-b(1+b) & ab-c(1+c) \\ ac-b(1+b) & (1+a)(1+b)-c^2 & bc-a(1+a) \\ ab-c(1+c) & bc-a(1+a) & (1+a)(1+c)-b^2 \end{pmatrix} \tag{101}$$

where

$$\Delta=\big(1+(a+b+c)\big)\left(1-\frac{1}{2}\big((a-b)^2+(b-c)^2+(a-c)^2\big)\right). \tag{102}$$

To check that $\hat{M}(0)$, $\hat{M}(1)$, and $\hat{M}(2)$ are invertible (which should be the case by Theorem 2), we need only show that $\Delta \neq 0$ in each case.

In the case of $\hat{M}(0)$, we have

$$a=\frac{9+3\sqrt{5}}{16}, \qquad b=\frac{14}{16}, \qquad c=\frac{9-3\sqrt{5}}{16} \tag{103}$$

which yields

$$\Delta = \frac{9}{8}. \tag{104}$$

In the case of $\hat{M}(1)$,

$$a = \frac{3+\sqrt{5}}{32}, \qquad b = -\frac{22}{32}, \qquad c = \frac{3-\sqrt{5}}{32} \tag{105}$$

with the result that

$$\Delta = \frac{3}{16}. \tag{106}$$

Finally, in the case of $\hat{M}(2)$,

$$a = -\frac{3+\sqrt{5}}{8}, \qquad b = \frac{2}{8}, \qquad c = -\frac{3-\sqrt{5}}{8} \tag{107}$$

from which we calculate

$$\Delta = \frac{3}{16}. \tag{108}$$

We now have the following concrete procedure for solving Eq. (58).
Given $\mathbf{C}(p)$, evaluate

$$\hat{\mathbf{C}}(h) = \sum_{p=-2}^{2} \mathbf{C}(p) \exp\left(-i\frac{2\pi}{5}ph\right) \tag{109}$$

for $h = 0, 1, 2$. Next, solve the 3×3 linear systems

$$\hat{M}(h)\hat{\mathbf{S}}(h) = \hat{\mathbf{C}}(h) \tag{110}$$

for $\hat{\mathbf{S}}(h)$ in each of the three cases $h = 0, 1, 2$. This can be done using the explicit formula for $(\hat{M}(h))^{-1}$ that was derived above. Then set $\hat{\mathbf{S}}(-h) = \text{conjugate}(\hat{\mathbf{S}}(h))$ for $h = 1, 2$, and finally evaluate $\mathbf{S}(q)$ according to

$$\mathbf{S}(q) = \frac{1}{5} \sum_{h=-2}^{2} \hat{\mathbf{S}}(h) \exp\left(i\frac{2\pi}{5}qh\right). \tag{111}$$

We emphasize that the above procedure can be carried out at each point $\mathbf{x}$ independently in the case of an inhomogeneous elasticity tensor. Since the procedure of finding the S_n from the elasticity tensor C_{ijkl} is a fixed linear operation at each point, $S_n(\mathbf{x})$ inherits the differentiability properties of the elasticity tensor.

It is instructive to consider the special case of a homogeneous isotropic material. The elastic energy density of such a material under small deformations may be written in the form

$$w = \frac{1}{2}(\lambda e_{ii} e_{jj} + \mu e_{ij} e_{ij}). \tag{112}$$

It follows that

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{113}$$

Note that this respects all of the symmetries that are required of an elasticity tensor (Eqs. (3)–(5)), but also that it is only totally symmetric in the special case $\lambda = \mu$.

We use the decomposition Eq. (23) of Theorem 1 to find a totally symmetric $\tilde{C}$ that is equivalent to C in the sense that it gives the same elastic energy for any small incompressible deformation, i.e., for any strain matrix e such that $\operatorname{trace}(e) = 0$. The construction of $\tilde{C}$, according to the remark at the end of Theorem 1, is as follows:

$$\begin{aligned}\tilde{C}_{ijkl} &= C_{ijkl} - b_{ij}\delta_{kl}\delta_{ij}b_{kl} \\ &= \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - b_{ij}\delta_{kl} - \delta_{ij}b_{kl}\end{aligned} \tag{114}$$

where b is computed from C according to the recipe given by Eqs. (28)–(29). The result of applying this recipe in the present case is

$$b_{ij} = \frac{1}{2}(\lambda - \mu)\delta_{ij} \tag{115}$$

so

$$\tilde{C}_{ijkl} = \mu(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{116}$$

which is indeed totally symmetric, and moreover is the special case of C_{ijkl} corresponding to $\lambda = \mu$, so it, too, represents an isotropic material.

As a check, note that

$$\begin{aligned}C_{ijkl}e_{ij}e_{kl} &= \big(\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\big)e_{ij}e_{kl} \\ &= \lambda e_{ii}e_{kk} + \mu(e_{kj}e_{kj} + e_{lj}e_{jl}) \\ &= \lambda e_{ii}e_{kk} + 2\mu e_{jk}e_{jk}\end{aligned} \tag{117}$$

whereas

$$\begin{aligned}\tilde{C}_{ijkl}e_{ij}e_{kl} &= \mu(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})e_{ij}e_{kl} \\ &= \mu e_{ij}e_{kk} + \mu(e_{jk}e_{jk} + e_{jl}e_{jl}) \\ &= \mu e_{jj}e_{kk} + 2\mu e_{jk}e_{jk}\end{aligned} \tag{118}$$

and these results are indeed equivalent when $\operatorname{trace}(e) = 0$ (even though they are otherwise not equivalent, except in the special case $\lambda = \mu$).

We now proceed to calculate the corresponding stiffnesses of the 15 families of fibers that will generate an elasticity tensor of the form given by Eq. (116). The first step is to evaluate

$$\tilde{C}^m = p_{\tilde{C}}\big(\mathbf{u}^m\big) \tag{119}$$

for each of the 15 directions $\mathbf{u}^m$. Let us first compute the polynomial $p_{\tilde{C}}$:

$$p_{\tilde{C}}(\mathbf{t}) = \mu(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})t_it_jt_kt_l = 3\mu\big(t_1^2 + t_2^2 + t_3^2\big)^2 = 3\mu|\mathbf{t}|^4. \tag{120}$$

This result conforms to our intuition that $\tilde{C}$ is the isotropic tensor; its corresponding polynomial only depends on the length of $\mathbf{t}$. Thus,

$$\tilde{C}^m = p_{\tilde{C}}\big(\mathbf{u}^m\big) = 3\mu\big|\mathbf{u}^m\big|^4 = 3\mu, \tag{121}$$

independent of m, since $|\mathbf{u}^m| = 1$ for all m.

Following the procedure outlined above,

$$\hat{\tilde{\mathbf{C}}}(h) = 3\mu\begin{pmatrix}1\\1\\1\end{pmatrix}\sum_{p=-2}^{2}\exp\left(-i\frac{2\pi}{5}ph\right) = 15\mu\begin{pmatrix}1\\1\\1\end{pmatrix}\delta_{h0}. \tag{122}$$

It follows that

$$\hat{\mathbf{S}}(h) = 15\mu\delta_{h0}\big(\hat{M}(0)\big)^{-1}\begin{pmatrix}1\\1\\1\end{pmatrix}. \tag{123}$$

It is clear, however, by inspection of $\hat{M}(0)$, see Eq (97), that $(1,1,1)^T$ is an eigenvector of $\hat{M}(0)$ with eigenvalue 3. Therefore,

$$\hat{\mathbf{S}}(h) = 5\mu\delta_{h0}\begin{pmatrix}1\\1\\1\end{pmatrix}. \tag{124}$$

Substituting this result into Eq. (111), we see that

$$\mathbf{S}(q) = \mu\begin{pmatrix}1\\1\\1\end{pmatrix} \tag{125}$$

for $q = -2, -1, 0, 1, 2$. In other words, all of the 15 fiber stiffnesses are equal to μ.

That all of the fiber stiffness coefficients come out equal in the representation of an isotropic material is no surprise; it merely confirms the symmetry of the particular fiber architecture we have chosen. What is somewhat more surprising, though, is that we are able to represent an isotropic elastic material by using only a *finite* number of directions for the fibers that run though any given point of the material. From a geometrical point of view, such a fiber architecture is *not* isotropic. It has 15 distinguished directions along which all of the fibers are oriented, and infinitely many other directions along which no fibers are oriented. Nevertheless, it behaves like an isotropic material under all possible (small) deformations. Indeed there is no (small) strain that one can apply that makes it possible to distinguish the fiber directions by a mechanical measurement (e.g., of stress or elastic energy).

This result, that 15 families of fibers suffice to simulate an isotropic material, has nothing to do with the constraint of incompressibility, since the fibers themselves have the elasticity tensor $\tilde{C}$ given by Eq. (116), and since this is the elasticity tensor of a particular isotropic material (one for which $\lambda = \mu$). Where the constraint of incompressibility is needed is if we want the 15 families of fibers to be able to simulate an *arbitrary* isotropic material (as in the present section), or more generally an *arbitrary* inhomogeneous anisotropic material (as in the rest of the present paper). In these cases, the restriction to consideration of (small) incompressible deformations provides just enough of a constraint to make our fiber architecture universal, as discussed at the end of the previous section.

Again, to avoid misunderstanding, we emphasize that the fibers themselves do not bear the burden of incompressibility. In the isotropic case, the Poisson ratio of the totally symmetric elasticity tensor that we have constructed, Eq. (116), is not 0.5. Our assumption is that the constraint of incompressibility is imposed upon the fibers by an incompressible medium in which they are embedded.

In the isotropic case, all of the fiber stiffness coefficients come out positive (and equal to each other). This implies, by continuity, that there is a neighborhood of the isotropic case in the space of elasticity tensors in which the corresponding fiber stiffnesses are all positive. We cannot, however, exclude the possibility that there exist elasticity tensors of stable incompressible materials for which the fiber representation constructed in this paper involves some fibers of negative stiffness.

In this connection, we note the related problem of finding the *minimal* number of fiber systems needed to represent a given *spatially homogeneous* totally symmetric tensor. With the identification of symmetric tensors with homogeneous polynomials, this problem is equivalent to writing a given $p \in \mathcal{P}_{34}$ as a linear combination of as few real coefficient linear forms as possible. It is known that at most six linear forms are needed and the coefficients of the linear combination (the fiber strength) are non-negative, so long as the associated quadratic form (which in our language corresponds to the elastic energy) is positive semidefinite. See [16] for a proof. For example, the isotropic tensor studied here can be represented with a six fiber system of equal strength in which the fiber directions are taken parallel to the lines connecting the antipodal vertices of the icosahedron [16].

We note, however, that the proof of the above result being of a non-constructive nature, it is computationally non-trivial to find explicitly such a minimal fiber representation for a given totally symmetric tensor [6]. This would be much more problematic in the setting of spatially inhomogeneous elasticity, in which case the six fiber direction field would have to be solved for at each point in space. To the best of our knowledge, there is even no guarantee that this direction field or the fiber stiffnesses will be devoid of singularities. In contrast, all we have to do with the 15 fiber system above is to solve a linear equation at each point, and the fiber stiffness coefficients inherit the smoothness properties of the original totally symmetric tensor. The price we pay is that we may have some negative fiber stiffnesses.

We note that the existence of negative fiber stiffness coefficients does *not* imply that the fiber system is unstable. The 15 families of fibers are coupled together by the requirement of continuity, that is, they all deform in accord with the deformation of the incompressible medium in which they are embedded. If we start with a stable incompressible linearly elastic material, i.e., with one whose elasticity tensor is a positive definite quadratic form when restricted to the space of incompressible deformations, then the fiber representation of that material is also stable, for the simple reason that its elastic energy, by definition, is exactly the same as that of the given material when both are subjected to the same incompressible deformation. This is true even in the case that some of the fiber stiffnesses may happen to be negative. Those negative stiffnesses, if they occur, are necessarily more than compensated by positive stiffnesses of other fiber families in such a way that arbitrary incompressible deformations result in positive energies. We do add, however, that this may pose significant challenges if we are to engineer such a material in physical reality. See [18,17] for an introduction to materials with negative stiffness and to [8] for an extensive reference.

6. Optimality of the icosahedral configuration

As we saw in Section 4, there are numerous ways in which to choose a 15 vector configuration for purposes of constructing a fiber representation. Theorem 2 states that symmetry considerations single out the icosahedral configuration. In the previous section, we found that the symmetry of the icosahedron allows us to construct an explicit algorithm to solve for the fiber strengths. In this section, we claim, by way of numerical evidence, that the icosahedral configuration satisfies two optimality properties within a certain class of configurations.

Inspired by Proposition 1, we ask for an optimum within the class $\mathcal{M}$ of configurations that can be generated as intersection lines of six planes. We shall formulate two conjectures, both of which say that the icosahedral configuration is optimal within the class $\mathcal{M}$. In both cases we have found examples which show that the restriction to the class $\mathcal{M}$ is essential, i.e., that the icosahedral configuration is not optimal by the criteria stated in the conjectures if arbitrary sets of 15 directions are considered.

Consider Eq. (58). It is natural to ask for a 15 vector configuration that minimizes the 2-norm condition number of U, which we shall denote by $\kappa(U)$.

Conjecture 1. *The unique minimum of $\kappa(U)$ over $\mathcal{M}$ is attained at the icosahedral configuration.*

We performed computations to test this conjecture, according to the following procedure. First, we randomly sample 10^4 six-unit-vector configurations, where each of the six unit vectors defines a plane. We compute the 15 unit vectors generated by the six planes, and compute $\kappa(U)$ for each case. We then take the configuration with the minimum $\kappa(U)$, and use this as the initial configuration for a Nelder–Mead optimization routine [15]. We restarted the Nelder–Mead optimization procedure 20 times to ensure that a local minimum was reached. The algorithm was implemented in Matlab.

We performed this procedure 100 times. The smallest value of κ attained and its corresponding normal vectors came to within 14 digits of the icosahedral values.

To show that the restriction to the class $\mathcal{M}$ is necessary for the above conjecture to be valid, we also tried choosing the 15 vectors arbitrarily. In this case we found that we could obtain values of κ smaller than the icosahedral value $\kappa^{\text{icosa}} = 8 + 2\sqrt{10} = 14.324555\ldots$. The smallest value attained with 15 unrestricted directions was about 12.35. We could not find any obvious geometrical characterization of the minimum configuration.

Let us now consider another optimization problem. As noted at the end of the previous section, the fiber strengths are not all positive in general. Consider the convex cone $\mathcal{C}$ of linear elasticity tensors for which the fiber strengths turn out to be non-negative:

$$C_{ijkl} = \sum_{n=1}^{15} S_n \mathbf{u}_i^n \mathbf{u}_j^n \mathbf{u}_k^n \mathbf{u}_l^n, \quad S_n \geqslant 0. \tag{126}$$

We would like to maximize the proportion of totally symmetric tensors that fall within this convex cone. To measure the solid angle subtended by $\mathcal{C}$, we use the measure induced by the inner product we introduced in Section 4.

We shall work with homogeneous polynomials. Take an arbitrary orthonormal basis of polynomials $f^n \in \mathcal{P}_{34}$. Express the linear forms $(\mathbf{u}^n \cdot \mathbf{t})^4$ in terms of f^n, and denote the corresponding vectors as $\mathbf{q}^n \in \mathbb{R}^{15}$. That is to say, the mth component of $\mathbf{q}^n$ is given by

$$q_m^n = \langle (\mathbf{u}^n \cdot \mathbf{t})^4, f^m \rangle. \tag{127}$$

Note that $\mathbf{q}^n$ are unit vectors since the 2-norm of $(\mathbf{u}^n \cdot \mathbf{t})^4$ is 1. Let Q be the matrix whose column vectors are $\mathbf{q}^n$. Let the positive cone generated by $\mathbf{q}^n$ in $\mathbb{R}^{15}$ be $\mathcal{D}$:

$$\mathcal{D} = \left\{ \mathbf{x} \in \mathbb{R}^{15} \;\middle|\; \mathbf{x} = \sum_{n=1}^{15} \lambda_n \mathbf{q}^n, \ \lambda_n \geqslant 0 \right\}. \tag{128}$$

Then the solid angle in question can be expressed as follows:

$$\omega = \int_{\mathcal{D} \cap S^{14}} d\Omega = 15 \int_{\mathcal{D} \cap \{\|\mathbf{x} \leqslant 1\|\}} d\mathbf{x} \tag{129}$$

where S^{14} is the 14-dimensional unit sphere, $\|\cdot\|$ denotes the standard Euclidean norm in $\mathbb{R}^{15}$, $d\Omega$ denotes integration with respect to solid angle and $d\mathbf{x}$ standard integration in $\mathbb{R}^{15}$. Let $\mathbb{R}_+^{15}$ be the positive cone whose components are positive in $\mathbb{R}^{15}$. The matrix Q maps $\mathbb{R}_+^{15}$ to $\mathcal{D}$. Thus,

$$\omega = 15 \int_{\mathcal{D} \cap S^{14}} d\mathbf{x} = 15 \int_{\mathbb{R}_+^{15} \cap \{\|Q\mathbf{y}\| \leqslant 1\}} \left|\det(Q)\right| d\mathbf{y}$$

$$= 15\left|\det(Q)\right| \int\limits_{\mathbb{R}_+^{15}\cap S^{14}} \int\limits_0^{\|Q\hat{\mathbf{y}}\|^{-1}} r^{14}\,dr\,d\Omega$$

$$= \left|\det(Q)\right| \int\limits_{\mathbb{R}_+^{15}\cap S^{14}} \|Q\hat{\mathbf{y}}\|^{-15}\,d\Omega \tag{130}$$

where $\hat{\mathbf{y}}$ varies over $\mathbb{R}_+^{15} \cap S^{14}$. As can be easily checked, $U = Q^T Q$, where Q^T is the transpose of Q. Thus,

$$\omega(U) = \sqrt{\det(U)} \int\limits_{\mathbb{R}_+^{15}\cap S^{14}} (\hat{\mathbf{y}}, U\hat{\mathbf{y}})^{-15/2}\,d\Omega \tag{131}$$

where $(\cdot,\cdot)$ denotes the Euclidean inner product in $\mathbb{R}^{15}$ and we have made explicit the dependence of ω on U. We thus seek to maximize this solid angle U over $\mathcal{M}$.

Conjecture 2. *The unique maximum of $\omega(U)$ over $\mathcal{M}$ is attained at the icosahedral configuration.*

Again, the restriction to $\mathcal{M}$ is necessary: computation shows that the icosahedral configuration is not the optimal by this criterion when the 15 directions are varied arbitrarily.

To evaluate (131) we use a Monte Carlo algorithm [15]. We uniformly sample $N^{\text{int}} = 10^4$ points on $\mathbb{R}_+^{15} \cap S^{14}$ and use the same sample points throughout one optimization run. Otherwise, the optimization algorithm is identical to the case of the condition number minimization.

Denote the Monte Carlo approximation of ω as ω^{MC}. Since we do not have the exact value of ω at the icosahedral configuration U^{icosa}, we compare the computed optimum to $\omega^{\text{MC}}(U^{\text{icosa}})$. The maximum values and the corresponding configuration came to within 4 digits of the icosahedral configuration.

7. Conclusions

Within the framework of incompressible linear elasticity, we have devised a fiber architecture that is universal in the sense that it can be used to simulate an arbitrary (possibly inhomogeneous) material by the appropriate choice of (possibly position-dependent) stiffness coefficients of the fibers.

This fiber architecture takes the form of 15 families of fibers. All of the fibers run along straight lines, and within any one family all of the fibers are parallel. Thus, the whole structure is determined by the choice of 15 unit vectors, each of which gives the direction of the fibers in a particular family.

The particular fiber architecture we propose is the most symmetric possible, obtained by connecting the antipodal midpoints of the edges of the icosahedron (which give the same directions as connecting the antipodal midpoints of the edges of the dodecahedron, by duality). There are 30 such midpoints, in 15 antipodal pairs.

To show that the fiber architecture constructed in this manner is universal, in the sense defined above, we prove two theorems. The first theorem states that an arbitrary elasticity tensor can be reduced, within the framework of incompressible linear elasticity, to an equivalent elasticity tensor that is totally symmetric. The second theorem states that a material with a totally symmetric elasticity tensor has a unique representation in terms of the 15-family fiber architecture described above, and furthermore, that this fiber architecture is the most symmetric possible in the sense that its symmetry group acts transitively on the set of fiber directions. In the course of this proof, we identify a large class of fiber architectures that are universal, namely those that are generated as intersection lines of 6 planes. The above icosahedral fiber architecture is a particular instance of this.

We also introduce an explicit procedure that can be used to solve for the stiffness coefficients of the representation. This procedure involves the discrete Fourier transform of order 5, which takes advantage of the fact that the 15 icosahedral directions come in 5 orthonormal triads.

Finally, we present computational evidence which suggests that this fiber architecture satisfies optimality properties within the class of fiber architectures generated by 6 planes.

The overall conclusion of this work is that the notion of a "fiber-reinforced fluid" is much more general than it seems. Indeed, we have shown how it is possible to construct a particular fiber-reinforced fluid that can simulate any incompressible linearly elastic material, merely by adjusting, or programming, the stiffness coefficients of the fibers.

An important question that remains is the physical realizability of our mathematical notion of "fiber-reinforced fluid." Although our notion is useful from a conceptual and computational point of view, the correspondence between our mathematical notion and the physical picture of fibers embedded in fluid remains to be clarified. A detailed understanding of this gap is likely to be essential in understanding the mechanical properties of biological materials. Indeed, the major mechanical supports of a biological cell come in the form of fibers, all of which are embedded in a cytosolic fluid.

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Appendix A. Generalization of Proposition 1

We now generalize Proposition 1 to totally symmetric tensors of arbitrary order and dimension. The set of totally symmetric tensors of order m in d dimensions can be identified with the set of homogeneous polynomials of order m in d variables, which we denote by $\mathcal{P}_{dm}$. The space $\mathcal{P}_{dm}$ has dimension $\binom{d+m-1}{d-1} = \frac{(d+m-1)!}{(d-1)!m!}$, as can be seen by a simple combinatorial argument. We may introduce an inner product on $\mathcal{P}_{dm}$ analogously to the case of $\mathcal{P}_{34}$. Consider the problem of finding a set of basis vectors of $\mathcal{P}_{dm}$ all of which are linear forms:

$$\left(\mathbf{u}^\alpha \cdot \mathbf{t}\right)^m, \quad \mathbf{u} \in \mathbb{R}^d,\ 1 \leqslant \alpha \leqslant \begin{pmatrix} d+m-1 \\ d-1 \end{pmatrix}. \tag{132}$$

We call such vectors $\mathbf{u}^\alpha$ a (d,m)-basis. We have the equivalent of Lemma 1 hold in $\mathcal{P}_{dm}$. We can now state our generalization.

Take $d+m-1$ hyperplanes $\pi_\alpha \subset \mathbb{R}^d$, $1 \leqslant \alpha \leqslant d+m-1$, whose unit normal vectors we denote by $\mathbf{n}^\alpha$, $1 \leqslant \alpha \leqslant d+m-1$. We say that these hyperplanes are in general position if any choice of d normal vectors is linearly independent.

Proposition 3. *Take hyperplanes π_α, $1 \leqslant \alpha \leqslant d+m-1$, in general position. The intersection of $d-1$ hyperplanes generates a line. There are $\binom{d+m-1}{d-1}$ such lines, all of which are distinct. The corresponding unit direction vectors form a (d,m)-basis.*

Proof. Let $\mathbf{u}^\alpha$, $1 \leqslant \alpha \leqslant \binom{d+m-1}{d-1}$, be a (d,m)-basis. For $p \in \mathcal{P}_{dm}$, we must show that $p(\mathbf{u}^\alpha) = 0$ for all α implies $p(\mathbf{z}) = 0$ for all $\mathbf{z} \in \mathbb{R}^d$.

Consider the subspaces generated by choosing $d-k$, $1 \leqslant k \leqslant d-2$, hyperplanes from π_α and taking their intersection. We call the set of all such subspaces Σ^k. Since the hyperplanes are in general position, one can easily see that a choice of $d-k$ hyperplanes defines a unique k-dimensional subspace. The set Σ^k thus consists of $\binom{d+m-1}{d-k}$ distinct linear subspaces of dimension k. We shall set Σ^{d-1} to be the set of the $d+m-1$ hyperplanes and Σ^d to be $\mathbb{R}^d$ for convenience.

We proceed by mathematical induction on k. Let $k=1$. If $\mathbf{z}$ lies on any of the subspaces of Σ^1 (that is, the intersection lines), $p(\mathbf{z})=0$ by assumption. Let $2 \leqslant k \leqslant d$. Suppose $p(\mathbf{z})=0$ for all $\mathbf{z}$ that lies on any of the subspaces in Σ^{k-1}. Take any $\sigma \in \Sigma^k$. Without loss of generality, assume that σ is the intersection of $\pi_1, \ldots, \pi_{d-k}$ (for $k=d$, this would be the empty set). By taking the intersection of σ with $\pi_{d-k+1}, \ldots, \pi_{d+m-1}$, we see that there are $m+k-1$ members of Σ^{k-1} contained in σ, which we denote by σ_l, $1 \leqslant l \leqslant m+k-1$. Take any point $\mathbf{z}$ in σ and consider a line $\ell \subset \sigma$ that intersects all of the subspaces σ_l. Such a line can be found since the subspaces σ_l have dimension $k-1$. Let these intersection points be z_i, $1 \leqslant i \leqslant m+k-1$. Consider the polynomial p restricted to ℓ. By the induction hypothesis, p is 0 at all z_i. This means that p has $m+k-1 \geqslant m+1$ roots along ℓ. Since p is a polynomial of order m, it must vanish, which means in particular that $p(\mathbf{z})=0$. $\square$

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